

Boundary Value Problems for some Fully Nonlinear Elliptic Equations

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Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ with boundary ∂M . We denote the Ricci curvature, scalar curvature, mean curvature, and the second fundamental form by Ric , R , h , and $L_{\alpha\beta}$, respectively.

The Yamabe problem for manifolds with boundary is to find a conformal metric $\hat{g} = e^{-2u}g$ such that the scalar curvature is constant and the mean curvature is zero. The boundary is called umbilic if the second fundamental form $L_{\alpha\beta} = \mu_g g_{\alpha\beta}$. For example, a totally geodesic boundary is umbilic with zero principal curvatures. In [8], it was proved by Escobar that for locally conformally flat compact manifolds with umbilic boundary (and some other cases), the Yamabe problem is solvable.

As for the nonlinear version of the Yamabe problem, we consider the Schouten tensor defined as

$$A_g = \frac{1}{n-2} \left(\text{Ric} - \frac{R}{2(n-1)} g \right).$$

Note that $\text{tr} A_g = \frac{1}{2(n-1)} R$. The Schouten tensor comes naturally from curvature decomposition

$$\text{Riem} = \mathcal{W} + A \odot g,$$

where the Weyl tensor \mathcal{W} is locally conformally invariant, and \odot stands for the Kulkarni-Nomizu product. In dimension four, we have the following Chern-Gauss-Bonnet formula for closed manifolds:

$$32\pi^2 \chi(M^4) = \int_{M^4} |\mathcal{W}|^2 + 16 \int_{M^4} \sigma_2(A_g),$$

where χ is the Euler characteristic and $\sigma_2(A_g)$ is the second elementary symmetric function of the eigenvalues of A_g . Since χ is a topological invariant and \mathcal{W} is locally conformally invariant, we have that $\int_M \sigma_2(A_g)$ is a conformal invariant. For closed four-manifolds, Chang-Gursky-Yang [5] proved that if the Yamabe constant and $\int_M \sigma_2(A_g)$ are both positive, then we can find a conformal metric \hat{g} such that $\sigma_2(A_{\hat{g}})$ is constant. For locally conformally flat closed manifolds, Li-Li [16] proved that if $\sigma_i(A_g) > 0$, $1 \leq i \leq k$ for some $k \geq 2$, then we can find a conformal metric \hat{g} such that $\sigma_k(A_{\hat{g}})$ is

constant. See also Guan-Wang [13] for an independent work of the above result. For closed manifolds which are not locally conformally flat, Gursky-Viaclovsky [14] proved that if $\sigma_i(A_g) > 0$, for $1 \leq i \leq k$ and $2k > n$, then we can find a conformal metric \hat{g} such that $\sigma_k(A_{\hat{g}})$ is constant.

In this paper, we study the nonlinear version of Yamabe problem for manifolds with boundary. Before introducing the problem, we need the following definitions:

Definition 1. Let W be a matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Then $\sigma_k(W) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k}$ for $k \leq n$ is called the k th elementary symmetric function of the eigenvalues of W . Denote $\sigma_0 = 1$. For example, $\sigma_1 = \lambda_1 + \dots + \lambda_n = \text{tr } W$ and $\sigma_n = \lambda_1 \dots \lambda_n = \det W$.

The elementary symmetric functions are special cases of hyperbolic polynomials introduced by Garding [10], which have nice properties in associated cones.

Definition 2. The set $\Gamma_k^+ = \{ \text{the connected component of } \sigma_k(\lambda) > 0 \text{ which contains the identity} \}$ is called the positive k -cone. Equivalently, it is showed in [10] that $\Gamma_k^+ = \{ \lambda : \sigma_i(\lambda) > 0, 1 \leq i \leq k \}$ is an open convex cone with vertex at the origin, e.g., $\Gamma_1^+ = \{ \lambda : \lambda_1 + \dots + \lambda_n > 0 \}$ and $\Gamma_n^+ = \{ \lambda : \lambda_i > 0, 1 \leq i \leq n \}$. The following is the nested relation

$$\Gamma_1^+ \supset \Gamma_2^+ \supset \dots \supset \Gamma_n^+.$$

Denote $W \in \Gamma_k^+$ if the eigenvalues $\lambda(W) \in \Gamma_k^+$.

Suppose that the boundary is umbilic. Our goal is to find a conformal metric $\hat{g} = e^{-2u}g$ such that $\sigma_k(A_{\hat{g}})$ is constant and the boundary is totally geodesic. We now describe a class of locally conformally flat compact manifolds of dimension $n \geq 3$ with boundary, for which we give an affirmative answer to the question. Under the conformal change of the metric $\hat{g} = e^{-2u}g$, we denote the curvature tensors in the new metric by a *hat* (For example, \hat{A} , \hat{L} and $\hat{\mu}$). The Schouten tensor \hat{A} satisfies

$$\hat{A} = \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g + A_g, \quad (1)$$

where the derivatives are covariant derivatives with respect to the background metric g . The second fundamental form satisfies

$$\hat{L}e^u = \frac{\partial u}{\partial n} g + L_g,$$

where n is the unit inner normal with respect to g on the boundary. Note that umbilicity is conformally invariant. Thus, it is natural to consider the class of manifolds with umbilic boundary. When the boundary is umbilic, the above formula becomes

$$\hat{\mu}e^{-u} = \frac{\partial u}{\partial n} + \mu_g.$$

If we view \hat{A} as a $(0, 2)$ -tensor in the new metric \hat{g} , then $\sigma_k(\hat{A}) := \sigma_k(\hat{g}^{-1}\hat{A})$, where \hat{g}^{-1} is the induced inverse tensor of the metric tensor \hat{g} . On the other hand, by formula (1) we can also view \hat{A} as a $(0, 2)$ -tensor in the background metric g . Using this notation, the problem becomes to consider the following equation:

$$\begin{cases} \sigma_k^{\frac{1}{k}}(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g) = e^{-2u} & \text{in } M \\ \frac{\partial u}{\partial n} + \mu_g = 0 & \text{on } \partial M. \end{cases} \quad (2)$$

Theorem 1. *Suppose (M, g) is a locally conformally flat compact manifold of dimension $n \geq 3$ with umbilic boundary. If $A_g \in \Gamma_k^+$ for $k \geq 2$, then there exists a smooth solution u of (2). In other words, there is a conformal metric $\hat{g} = e^{-2u}g$ such that $\sigma_k(\hat{A}) = 1$ and the boundary is totally geodesic.*

We will prove a more general result than Theorem 1. Consider the equation

$$\begin{cases} F(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g) = e^{-2u} & \text{in } M \\ \frac{\partial u}{\partial n} + \mu_g = 0 & \text{on } \partial M, \end{cases} \quad (3)$$

where F satisfies some structure conditions listed below. Equation (3) means that we apply F to the eigenvalues of the matrix (or $(1, 1)$ -tensor) $g^{-1}(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g)$. Now we give structure conditions for F . Let Γ be an open convex cone in \mathbb{R}^n with vertex at the origin satisfying $\Gamma_n^+ \subset \Gamma \subset \Gamma_1^+$. Suppose that $F(\lambda) = F(\sigma_1(\lambda), \dots, \sigma_n(\lambda)) \in C^\infty(\Gamma) \cap C^0(\bar{\Gamma})$ is a homogeneous symmetric function of degree one normalized with $F(e) = F(1, \dots, 1) = 1$. Assume that $F = 0$ on $\partial\Gamma$ and F satisfies the following in Γ :

- (S0) F is positive;
 - (S1) F is concave (i.e., $\frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j}$ is negative semi-definite);
 - (S2) F is monotone (i.e., $\frac{\partial F}{\partial \lambda_i}$ is positive);
 - (S3) $\frac{\partial F}{\partial \lambda_i} \geq \epsilon \frac{F}{\sigma_1}$, for some constant $\epsilon > 0$, for all i .
- In some case, we need an additional condition:
- (A) $\sum_{j \neq i} \frac{\partial F}{\partial \lambda_j} \leq \rho \frac{\partial F}{\partial \lambda_i}$, for some $\rho > 0$, for all $\lambda \in \Gamma$ with $\lambda_i \leq 0$.

An easy example is $F = \frac{1}{n}(\lambda_1 + \dots + \lambda_n)$ with $\Gamma = \{\lambda : \lambda_1 + \dots + \lambda_n > 0\}$. Condition (S1) is used in most elliptic theories. Condition (S2) is the actual ellipticity. It is an elementary fact that if F is a symmetric function of eigenvalues, then $\frac{\partial F}{\partial \lambda_i} > 0$ for all i if and only if $F^{ij} := \frac{\partial F}{\partial w_{ij}}$ is positive definite. Condition (S3) was before in [5].

Theorem 2. *Suppose (M, g) is a locally conformally flat compact manifold of dimension $n \geq 3$ with umbilic boundary. Let F satisfy the structure conditions (S0)-(S3) in a corresponding cone Γ . If $A_g \in \Gamma$, then there exists a smooth solution u of (3).*

In Section 1 below, we will show that $\binom{n}{k}^{-\frac{1}{k}} \sigma_k^{\frac{1}{k}}$ satisfies the structure conditions (S0)-(S3) with $\epsilon = \frac{1}{k}$ in Γ_k^+ . Hence, Theorem 2 implies Theorem 1.

The next result concerns boundary estimates for equations more general than (3). Before stating the theorem, we introduce some notations. In this paper, we use Fermi

(geodesic) coordinates in a boundary neighborhood, which means that we take the geodesics in the normal direction parameterized by arc length from a local chart (x_1, \dots, x_{n-1}) on the boundary. The metric is then expressed as $g = dx^n dx^n + g_{\alpha\beta} dx^\alpha dx^\beta$. The Greek letters α, β, γ stand for the tangential direction indices, $1 \leq \alpha, \beta, \gamma < n$, while the letters i, j, k stand for the full indices, $1 \leq i, j, k \leq n$. Define the half ball in Fermi coordinates by $\overline{B}_r^+ = \{x_n \geq 0, \sum_i x_i^2 \leq r^2\}$ and the segment on the boundary by $\Sigma_r = \{x_n = 0, \sum_i x_i^2 \leq r^2\}$. All derivatives are covariant derivatives with respect to the background metric g unless otherwise noted.

The following boundary estimates are used in the proof of Theorem 2.

Theorem 3. *Let F satisfy (S0)-(S3) in a corresponding cone Γ and g be a flat metric. Suppose that Σ_r is umbilic with principal curvatures μ and n is the unit inner normal with respect to g . Let $u \in C^4$ be a solution to the equation*

$$\begin{cases} F(\nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g) = f e^{-2u} & \text{in } \overline{B}_r^+ \\ \frac{\partial u}{\partial n} + \mu = \hat{\mu} e^{-u} & \text{on } \Sigma_r. \end{cases} \quad (4)$$

Case(a). If $\hat{\mu} = 0$, then

$$\sup_{x \in \overline{B}_{\frac{r}{2}}^+} (|\nabla u|^2 + |\nabla^2 u|) \leq C(1 + \sup_{x \in \overline{B}_r^+} e^{-2u}),$$

where C depends on $r, n, \epsilon, \mu, \|f\|_{C^2(\overline{B}_r^+)}$ and $\inf_{\overline{B}_r^+} f$.

Case(b). Suppose that F satisfies the additional condition (A) and $\Gamma_2^+ \subset \Gamma$. If $\hat{\mu}$ is a positive constant, then

$$\sup_{x \in \overline{B}_{\frac{r}{2}}^+} (|\nabla u|^2 + |\nabla^2 u|) \leq C,$$

where C depends on $r, n, \epsilon, \rho, \mu, \hat{\mu}, \inf_{\overline{B}_r^+} u, \|f\|_{C^2(\overline{B}_r^+)}$ and $\inf_{\overline{B}_r^+} f$.

In Section 1 below, we further show that $\binom{n}{k}^{-\frac{1}{k}} \sigma_k^{\frac{1}{k}}$ satisfies the additional condition (A) with $\rho = (n - k)$. Thus, $\binom{n}{k}^{-\frac{1}{k}} \sigma_k^{\frac{1}{k}}$ for $k \geq 2$ is an example of case (b).

The Dirichlet problems for fully nonlinear elliptic equations have been extensively studied, for example, by Caffarelli-Nirenberg-Spruck [2], [3] and by Trudinger [23]. Such problems for the Schouten tensor equations are studied by Guan [11]. On the other hand, the Neumann problems for fully nonlinear elliptic equations are not yet well studied. The problem we proposed here comes from natural geometrical setting. It would be an interesting problem whether we can consider other Monge-Ampere-type equations.

The idea of proof of Theorem 2 is to deform the Yamabe metric for manifolds with boundary to the one satisfying the equation (3). The similar idea has already appeared in [16] and [15] for closed manifolds. We will show that, to avoid the bubbling phenomenon, if a manifold is not conformally equivalent to hemispheres, we have a priori estimates. Hence by degree theory argument we obtain a solution. The proof of boundary C^0 estimates follows closely that of Li-Li [16], while we still need to prove a

revised version of the work by Schoen-Yau [21], which turns out to be a crucial element. As for C^2 estimates, local C^2 estimates are previously proved by Chang-Gursky-Yang [4], Guan-Wang [12] and Li-Li [16] in different cases. Recently, a simplified proof of local C^2 estimates is derived by Chen [7] and applied to a large class of equations. To prove Theorem 3, we will use an idea in that work to derive boundary C^2 estimates directly from boundary C^0 estimates, which is the main part of this paper.

The above results extend to manifolds with boundary which are not locally conformally flat. In a subsequent paper [6], we study boundary value problems associated to some integral invariants on manifolds with boundary.

The paper is organized as follows. We start with some background in Section 1. In Sections 2 and 3, we give the proofs of Theorem 2 and 3, respectively.

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1 Background

We give some basic facts about homogeneous symmetric functions.

Lemma 1. (see [7]). *Let Γ be an open convex cone with vertex at the origin satisfying $\Gamma_n^+ \subset \Gamma$, and let $e = (1, \dots, 1)$ be the identity. Suppose that F is a homogeneous symmetric function of degree one normalized with $F(e) = 1$, and that F is concave in Γ . Then*

$$(a) \sum_i \lambda_i \frac{\partial F(\lambda)}{\partial \lambda_i} = F(\lambda), \quad \text{for } \lambda \in \Gamma.$$

$$(b) \sum_i \frac{\partial F(\lambda)}{\partial \lambda_i} \geq F(e) = 1, \quad \text{for } \lambda \in \Gamma.$$

Now we list further properties of elementary symmetric functions.

Lemma 2. (see [10], [19] and [3]). *Let $G = \sigma_k^{\frac{1}{k}}, k \leq n$. Then*

(a) G is positive and concave in Γ_k^+ .

(b) G is monotone in Γ_k^+ , i.e., the matrix $G^{ij} = \frac{\partial G}{\partial W_{ij}}$ is positive definite.

(c) For $0 \leq l < k \leq n$, the following is the Newton-MacLaurin inequality

$$k(n-l+1)\sigma_{l-1}\sigma_k \leq l(n-k+1)\sigma_l\sigma_{k-1}.$$

Therefore, $S = \binom{n}{k}^{-\frac{1}{k}} G$ satisfies the structure conditions (S0)-(S2) in Γ_k^+ .

We use the notation $\Lambda_i = (\lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_n)$. We will show that $S = \binom{n}{k}^{-\frac{1}{k}} G$ satisfies (S3) by using the following lemma:

Lemma 3. *Let $n \geq 2$. If $\lambda \in \Gamma_k^+$ for some $1 \leq k \leq n$, then*

$$\sigma_{k-1}(\Lambda_i) \geq \frac{\sigma_k(\lambda)}{\sigma_1(\lambda)} \quad \forall i.$$

Proof. Since $\lambda \in \Gamma_k^+$, we have $\frac{\partial \sigma_l}{\partial \lambda_i} = \sigma_{l-1}(\Lambda_i) > 0$, for $1 \leq l \leq k$, and thus $\Lambda_i \in \Gamma_{k-1}^+(\mathbb{R}^{n-1})$. On the other hand, by definition we have the identity $\sigma_{k-1}(\Lambda_i)\sigma_1(\lambda) = \sigma_{k-1}(\Lambda_i)\lambda_i + \sigma_{k-1}(\Lambda_i)\sigma_1(\Lambda_i)$.

Case (1): For $k = 1$, we get $\sigma_{k-1}(\Lambda_i) = 1 = \frac{\sigma_1(\lambda)}{\sigma_1(\lambda)}$.

Case (2): For $2 \leq k \leq n-1$, by Lemma 2 (C), $(n-k)\sigma_1(\Lambda_i)\sigma_{k-1}(\Lambda_i) \geq k(n-1)\sigma_k(\Lambda_i)$. If $\sigma_k(\Lambda_i) \geq 0$, then

$$\sigma_1(\Lambda_i)\sigma_{k-1}(\Lambda_i) \geq \frac{k(n-1)}{n-k}\sigma_k(\Lambda_i) \geq \sigma_k(\Lambda_i).$$

If $\sigma_k(\Lambda_i) < 0$, then

$$\sigma_1(\Lambda_i)\sigma_{k-1}(\Lambda_i) > 0 > \sigma_k(\Lambda_i).$$

Thus, in both cases, $\sigma_{k-1}(\Lambda_i)\sigma_1(\lambda) \geq \sigma_{k-1}(\Lambda_i)\lambda_i + \sigma_k(\Lambda_i) = \sigma_k(\lambda)$.

Case (3): For $k = n$, we have $\sigma_{n-1}(\Lambda_i)\sigma_1(\lambda) \geq \sigma_{n-1}(\Lambda_i)\lambda_i = \sigma_n(\lambda)$. \square

As a consequence of the above lemma, $S = \binom{n}{k}^{-\frac{1}{k}} \sigma_k^{\frac{1}{k}}$ satisfies (S3) with $\epsilon = \frac{1}{k}$.

The next lemma shows that $S = \binom{n}{k}^{-\frac{1}{k}} \sigma_k^{\frac{1}{k}}$ also satisfies the additional condition (A) with $\rho = (n-k)$.

Lemma 4. *For $1 \leq k \leq n-1$, if $\lambda \in \Gamma_k^+$ with $\lambda_i \leq 0$ for some i , then*

$$\sum_{j \neq i} \frac{\partial \sigma_k(\lambda)}{\partial \lambda_j} \leq (n-k) \frac{\partial \sigma_k(\lambda)}{\partial \lambda_i}.$$

Proof. For $k = 1$, the above inequality is trivial since $\frac{\partial \sigma_1(\lambda)}{\partial \lambda_j} = 1$ for all j . For $k \geq 2$, we have

$$\begin{aligned} \sum_j \frac{\partial \sigma_k(\lambda)}{\partial \lambda_j} &= (n-k+1)\sigma_{k-1}(\lambda) = (n-k+1)(\sigma_{k-1}(\Lambda_i) + \lambda_i\sigma_{k-2}(\Lambda_i)) \\ &\leq (n-k+1)\sigma_{k-1}(\Lambda_i) = (n-k+1) \frac{\partial \sigma_k(\lambda)}{\partial \lambda_i}. \end{aligned}$$

By cancelling out $\frac{\partial \sigma_k(\lambda)}{\partial \lambda_i}$ on both sides, the lemma is proved. \square

Suppose that F satisfies (S0)-(S3) in Γ . It is useful to consider the following symmetric functions, which are introduced in [16].

Definition 3. *Let $F^t(\lambda) = (t + n(1-t))^{-1}F(t\lambda + (1-t)\sigma_1(\lambda)e)$, for $0 \leq t \leq 1$ in the cone $\Gamma^t = \{\lambda : t\lambda + (1-t)\sigma_1(\lambda)e \in \Gamma\}$.*

We show that F^t satisfies (S0)-(S3) in Γ^t . It is easy to see that F^t is positive and concave. For monotonicity,

$$(t + n(1 - t)) \frac{\partial F^t}{\partial \lambda_i} = tF_i + (1 - t) \sum_j F_j \geq F_i > 0.$$

As for (S3),

$$\frac{\partial F^t}{\partial \lambda_i} \geq \epsilon \frac{F^t(\lambda)}{\sigma_1(t\lambda + (1 - t)\sigma_1(\lambda)e)}(t + n(1 - t)) = \epsilon \frac{F^t(\lambda)}{\sigma_1(\lambda)}.$$

Finally, if $F(\lambda) = F(\sigma_1(\lambda), \dots, \sigma_n(\lambda))$, then $F^t(\lambda)$ is a function of $\sigma_1(\lambda), \dots, \sigma_n(\lambda)$. This is because $\sigma_k(t\lambda + (1 - t)\sigma_1(\lambda)e)$, a homogeneous symmetric polynomial, is a function of $\sigma_1(\lambda), \dots, \sigma_n(\lambda)$ by elementary algebra.

The next lemma concerns some important behaviors of solutions on the boundary. As we mentioned in the introduction, in this paper we use Fermi coordinates in a boundary neighborhood. Before stating the lemma, we introduce a definition:

Definition 4. (see [20]). Let P be a symmetric matrix. $T_k = \sigma_k I - \sigma_{k-1}P + \dots + (-1)^k P^k$ is called the k -th Newton tensor associated with P . We have that $\frac{\partial \sigma_k(P)}{\partial P_{ij}} = (T_{k-1})_{ij}$.

Lemma 5. Let $F = F(\sigma_1(g^{-1}\hat{A}), \dots, \sigma_n(g^{-1}\hat{A}))$. Suppose g is flat and $\hat{L}_{\alpha\beta} = \hat{\mu}\hat{g}_{\alpha\beta}$ for some constant $\hat{\mu}$ near a boundary point x_0 . Then

- (a) $F^{\alpha n} = 0$ at x_0 ,
- (b) $\hat{A}_{\alpha\beta,n} = 2\mu\hat{A}_{\alpha\beta} - \hat{\mu}e^{-u}(\hat{A}_{\alpha\beta} + \hat{A}_{nn}g_{\alpha\beta})$ at x_0 .

Proof. Since g is flat, we have $\hat{A} = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g$. We denote the covariant differentiation with respect to the new metric \hat{g} by $\hat{\nabla}$. By the Codazzi equation

$$\hat{R}_{\alpha\beta\gamma n} = \hat{\nabla}_\beta \hat{L}_{\alpha\gamma} - \hat{\nabla}_\alpha \hat{L}_{\beta\gamma},$$

we have $\hat{R}_{\alpha\beta\gamma n} = 0$ because $\hat{\mu}$ is constant. Thus, we obtain $\hat{R}_{\alpha n} = 0$ and $\hat{A}_{\alpha n} = 0$ at x_0 . To prove (a), since F is a function of σ_i , we only need to show that $\frac{\partial \sigma_i(g^{-1}\hat{A})}{\partial \hat{A}_{\alpha n}} = (T_{i-1})_{\alpha n} = 0$ for all i . We prove it by induction. For $i = 1$, by definition $(T_1)_{\alpha n} = \sigma_1(g^{-1}\hat{A})g_{\alpha n} - \hat{A}_{\alpha n}$, which equals to zero. For general i , notice the recursive relation $(T_i)_{\alpha n} = \sigma_i(g^{-1}\hat{A})g_{\alpha n} - (T_{i-1})_{\alpha j}\hat{A}_{jn}$. Applying the induction hypothesis gives $(T_i)_{\alpha n} = -(T_{i-1})_{\alpha\beta}\hat{A}_{\beta n} = 0$.

For (b), note that the boundary is umbilic. Thus, u satisfies $\frac{\partial u}{\partial n} + \mu = \hat{\mu}e^{-u}$ on the boundary near x_0 . Since g is flat, by the Codazzi equation, μ is a constant. Notice that $\Gamma_{\alpha n}^n = 0$, $\Gamma_{\alpha\beta}^n = \mu g_{\alpha\beta}$ and $\Gamma_{\alpha n}^\beta = -\mu\delta_{\alpha\beta}$. Using the boundary condition, straightforward computations give us

$$u_{n\alpha} = -\hat{\mu}e^{-u}u_\alpha - \sum_j \Gamma_{\alpha n}^j u_j = \mu u_\alpha - \hat{\mu}u_\alpha e^{-u}, \quad (5)$$

and

$$\begin{aligned} u_{\alpha\beta n} &= (\mu - \hat{\mu}e^{-u})(u_{\alpha\beta} + \sum_j \Gamma_{\alpha\beta}^j u_j) + \hat{\mu}u_\alpha u_\beta e^{-u} - \sum_l \Gamma_{\beta n}^l u_{l\alpha} - \sum_l \Gamma_{\alpha\beta}^l u_{nl} \\ &= (2\mu - \hat{\mu}e^{-u})u_{\alpha\beta} - \mu u_{nn} g_{\alpha\beta} + \hat{\mu}u_\alpha u_\beta e^{-u} - \mu(-\mu + \hat{\mu}e^{-u})^2 g_{\alpha\beta}. \end{aligned} \quad (6)$$

Thus,

$$\begin{aligned} \hat{A}_{\alpha\beta, n} &= u_{\alpha\beta n} + u_{\alpha n} u_\beta + u_{\beta n} u_\alpha - \sum_l u_l u_{ln} g_{\alpha\beta} \\ &= 2\mu(u_{\alpha\beta} + u_\alpha u_\beta - \frac{1}{2}|\nabla u|^2 g_{\alpha\beta}) - \hat{\mu}e^{-u}(u_{\alpha\beta} + u_\alpha u_\beta + (-\sum_\gamma u_\gamma^2 + u_{nn})g_{\alpha\beta}), \end{aligned}$$

which equals to $2\mu\hat{A}_{\alpha\beta} - \hat{\mu}e^{-u}(\hat{A}_{\alpha\beta} + \hat{A}_{nn}g_{\alpha\beta})$. \square

Remark: In above lemma, (b) can be proved in an another way. Since g is flat, $\hat{\mathcal{W}}$ vanishes. Thus, by curvature decomposition \hat{R}_{ijkl} can be written in terms of \hat{R}_{ij} . Then using the Bianchi identity, we can compute $\hat{A}_{\alpha\beta, n}$.

2 Proof of Theorem 2

Proof. We deform the Yamabe metric to the one satisfying the equation (3). Define $F^t = (t + n(1-t))^{-1}F(t\lambda + (1-t)\sigma_1(\lambda)e)$ in Γ^t as in Section 1. Let the background metric g be the Yamabe metric such that R_g is a positive constant and the boundary is totally geodesic. Thus, the equation becomes the following:

$$\begin{cases} F^t(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A_g) = e^{-2u} & \text{in } M \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial M. \end{cases} \quad (7)$$

We will derive later a priori estimates for this path of equations for (M, g) not conformal equivalent to standard hemispheres (S_+^n, g_c) , where g_c is the standard metric on spheres. The Leray-Schauder degree is defined similarly as in Li [17]. In our case, we just consider the space $\{u \in C^{4,\alpha}(M) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial M\}$ instead of $\{u \in C^{4,\alpha}(M)\}$ for most closed manifolds cases. Then by homotopy-invariance we obtain a solution at $t = 1$, since at $t = 0$ the degree is nonzero. The fact that at $t = 0$ the degree is nonzero is proved by Schoen [22] for the Yamabe problem on closed manifolds. In our case, $\frac{\partial u}{\partial n} = 0$ on ∂M so the boundary integral terms vanish in the computations in [22]. Thus, the result remains the same. The problem then reduces to establishing a priori estimates.

Suppose F satisfies conditions (S0)-(S3). As in the discussion in Section 1, F^t also satisfies (S0)-(S3). We drop t without loss of generality in proving a priori estimates. We denote the conformal equivalence relation by \cong .

(1) C^0 estimates for $(M, g) \not\cong (S_+^n, g_c)$.

Since the boundary is totally geodesic, it is natural to consider the doubling of the manifold (M, g) and apply the C^0 estimates on locally conformally flat closed manifolds.

However, one problem is that we need to verify the doubling of the manifold still inherits a locally conformally flat smooth structure. Another problem is that the work by Schoen-Yau [21] is for locally conformally flat smooth manifolds, which is a crucial element in the proof of C^0 estimates. Thus, we need a revised version of that work for locally conformally flat $C^{2,\alpha}$ manifolds (or at least for the case of doubling of the manifold), which will be verified below. Then the rest of proof follows from that in [16] as we explain later.

Let (M^n, g) be a locally conformally flat compact manifolds with totally geodesic boundary. We denote a boundary neighborhood in M by $U_a \cup \partial' U_a$ where U_a is open and $\partial' U_a = \partial M \cap \partial U_a$ is a segment on the boundary. By definition, there is a conformal map $\phi_a : U_a \cup \partial' U_a \rightarrow V_a \cup \partial' V_a \subset S_+^n \cup S^{n-1}$ such that $V_a \subset S_+^n$ and $\partial' V_a$ is on the equator. Denote the doubling of M by $N = M \cup M^*$. We will define a locally conformally flat smooth structure on N . Define the corresponding conformal map ϕ_a^* from $U_a^* \subset M^*$ to $V_a^* \subset S_-^n$ through reflection. If ϕ_b and ϕ_b^* is another pair of conformal map such that $U_a \cap U_b$ (and thus $U_a^* \cap U_b^*$) is nonempty, then there is a conformal transformation Φ from $\phi_a(U_a \cap U_b)$ to $\phi_b(U_a \cap U_b)$. Similarly, there is a corresponding conformal transformation Φ^* on the counterpart. By Liouville theorem, the conformal transformations Φ and Φ^* can be extended to conformal transformations on S^n , still denoted by Φ and Φ^* . If we can prove that $\Phi = \Phi^*$, then they define a locally conformally flat smooth structure on N . Suppose that Φ and Φ^* are not equal. Then $\Phi^{-1} \circ \Phi^*$ is not the identity map on S^n . Notice that it is the identity map on $\phi_a(\partial'(U_a \cap U_b))$, which is a co-dimensional one submanifold contained in the equator. Thus, $\Phi^{-1} \circ \Phi^*$ must be a reflection with respect to the equator (see for example, Chapter A in [1]). This gives us an contradiction because Φ^{-1} can not map $\phi_b^*(U_a^* \cap U_b^*) \subset S_-^n$ to $\phi_a(U_a \cap U_b) \subset S_+^n$.

We still denote the metric extended to N by g . (N, g) is then a locally conformally flat closed manifold with $g \in C^{2,\alpha}$. We also have $(N, g) \not\cong (S^n, g_c)$ because $(M, g) \not\cong (S_+^n, g_c)$. Moreover, each side of differentiations in g is defined. We can follow the proof in [21] to show that there is a $C^{2,\alpha}$ developing map from the universal cover \tilde{N} to S^n . Note that each side of third derivatives in g is defined. Hence, the Liouville theorem is still valid since the proof is by an ordinary-differential-equations approach. Now that $R_g > 0$ on \tilde{N} , by the same argument as in [21], the developing map is injective. Solutions on M to (7) are extended naturally to the ones in $C^{2,\alpha}$ on \tilde{N} . To get C^0 bounds of u , the proof follows from that in [16] (proof of (1.44)) with some revise as we state below. First, instead of using Theorem 1.20 in [16], we use local estimates in [7] to drop the condition H_1 in establishing (4.1) in [16]. We also drop condition (1.41) in [16] by noting that the function F we consider is homogeneous, symmetric and normalized with $F(e) = 1$. After getting lower bounds of u on (M, g) (or equivalently upper bounds in [16] because the functions are chosen differently), by local estimates [7] and Theorem 3 we obtain the Harnack inequality

$$\max_M u \leq C \min_M u.$$

Thus, we only need to prove that $\min_M u$ is upper bounded. This follows from the fact that at the minimum point x_0 , we have $\hat{A} = \nabla^2 u + A_g \geq A_g$, where we use the boundary

condition $u_n = 0$ when x_0 is on the boundary. Therefore,

$$e^{-2\min_M u} = F(\nabla^2 u(x_0) + A_g(x_0)) \geq F(A_g(x_0)) > 0.$$

(2) C^2 estimates.

Interior C^2 estimates are proved in [7]. To get boundary C^2 estimates, we use Fermi coordinates in a tubular neighborhood $\partial M \times [0, \iota]$ of the boundary. Note that ∂M is compact so ι is a positive number. Since g is locally conformally flat, in a local chart we can choose a flat metric g_0 , which is conformal to g , such that μ_{g_0} is a constant and $\hat{\mu}$ is zero. Thus, by Theorem 3, we obtain boundary C^2 estimates in each half ball \overline{B}_r^+ . Since ∂M is compact, there are finitely many local charts of a tubular neighborhood of the boundary. We then get the desired estimates.

(3) C^∞ estimates.

Once we have C^2 bounds, F is uniformly elliptic and concave. By Evans-Krylov [9] and Lions-Trudinger [18], we have $C^{2,\alpha}$ estimates in the interior and on the boundary, respectively. Higher order regularity follows by standard elliptic theory. \square

3 Proof of Theorem 3

In this Section, we prove boundary estimates. We will use an idea in [7] to derive boundary C^2 estimates directly from boundary C^0 estimates.

Proof. Since g is flat, by Codazzi equation $\mu_g = \mu$ is constant on Σ_r . Let $\hat{A} = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g$. The condition $\Gamma_1^+ \subset \Gamma$ gives

$$0 < \sigma_1(\hat{A}) = \text{trace } \hat{A} = \Delta u - \frac{n-2}{2}|\nabla u|^2.$$

Thus, Δu is positive and

$$|\nabla u|^2 < C\Delta u. \quad (8)$$

(1) We show that u_{nnn} can be controlled on the boundary. Differentiating the equation on both sides in the normal direction at a boundary point, we get

$$(fe^{-2u})_n = \sum_{\alpha,\beta} F^{\alpha\beta} \hat{A}_{\alpha\beta,n} + F^{nn} \hat{A}_{nn,n},$$

where we have used $F^{\alpha n} = 0$ by Lemma 5.

For case (a), by Lemma 5 again $\hat{A}_{\alpha\beta,n} = 2\mu \hat{A}_{\alpha\beta}$. Thus,

$$\begin{aligned} (fe^{-2u})_n &= \sum_{\alpha,\beta} 2\mu F^{\alpha\beta} \hat{A}_{\alpha\beta} + F^{nn} \hat{A}_{nn,n} \\ &= 2\mu F + F^{nn}(\hat{A}_{nn,n} - 2\mu \hat{A}_{nn}) = 2\mu fe^{-2u} + F^{nn}(\hat{A}_{nn,n} - 2\mu \hat{A}_{nn}), \end{aligned} \quad (9)$$

where the second equality holds by Lemma 1 (a). By (5) and (6), we obtain

$$\begin{aligned}\hat{A}_{nn,n} - 2\mu\hat{A}_{nn} &= u_{nnn} + (u_n - 2\mu)u_{nn} - \sum_{\alpha} u_{\alpha}u_{\alpha n} - 2\mu(u_n^2 - \frac{1}{2}|\nabla u|^2) \\ &= u_{nnn} - 3\mu u_{nn} - \mu^3.\end{aligned}$$

Returning to (9), we get

$$-Ce^{-2u} \leq F^{nn}(\hat{A}_{nn,n} - 2\mu\hat{A}_{nn}) \leq F^{nn}(u_{nnn} - 3\mu u_{nn} + C).$$

On the other hand, by condition (S3) we have $F^{nn} \geq \epsilon \frac{F}{\sigma_1} \geq \frac{C}{\Delta u} e^{-2u}$. Hence, there is a positive number L such that

$$u_{nnn} \geq -L\Delta u + 3\mu u_{nn} - C \quad (10)$$

is true for every point on the boundary, where L and C depends on $n, \epsilon, \mu, \|f\|_{C^1}$ and $\inf f$.

For case (b), by Lemma 5 (b) we get

$$\begin{aligned}(fe^{-2u})_n &= \sum_{\alpha,\beta} F^{\alpha\beta}(2\mu\hat{A}_{\alpha\beta} - \hat{\mu}e^{-u}(\hat{A}_{\alpha\beta} + \hat{A}_{nn}g_{\alpha\beta})) + F^{nn}\hat{A}_{nn,n} \\ &= (2\mu - \hat{\mu}e^{-u})fe^{-2u} - \hat{\mu}e^{-u} \sum_{\alpha} F^{\alpha\alpha}\hat{A}_{nn} + F^{nn}(\hat{A}_{nn,n} - (2\mu - \hat{\mu}e^{-u})\hat{A}_{nn}),\end{aligned}$$

where the second equality holds by Lemma 1 (a). Note that $\hat{\mu}$ is positive. Thus, if $\hat{A}_{nn} \geq 0$, then

$$-Ce^{-2u} \leq F^{nn}(\hat{A}_{nn,n} - (2\mu - \hat{\mu}e^{-u})\hat{A}_{nn}).$$

On the other hand, if $\hat{A}_{nn} < 0$, by condition (A) we have

$$-Ce^{-2u} \leq F^{nn}(\hat{A}_{nn,n} - (2\mu + \rho\hat{\mu}e^{-u})\hat{A}_{nn}),$$

where we drop the term $F^{nn}\hat{\mu}e^{-u}\hat{A}_{nn}$ since it is negative. Hence, in both cases we obtain

$$-Ce^{-2u} \leq F^{nn}(\hat{A}_{nn,n} - 2\mu\hat{A}_{nn} + C|\hat{A}_{nn}|). \quad (11)$$

Now by (5) and (6) and combined with a basic fact that if $\Gamma_2^+ \subset \Gamma$, then $|u_{ij}| \leq C\Delta u$, we get

$$\hat{A}_{nn,n} - 2\mu\hat{A}_{nn} + C|\hat{A}_{nn}| \leq u_{nnn} + (-3\mu + \hat{\mu}e^{-u})u_{nn} + C\Delta u + C.$$

Returning to (11), note that by condition (S3) we have $F^{nn} \geq \epsilon \frac{F}{\sigma_1} \geq \frac{C}{\Delta u} e^{-2u}$. Hence, there is a positive number L such that

$$u_{nnn} \geq -L\Delta u + (3\mu - \hat{\mu}e^{-u})u_{nn} - C \quad (12)$$

is true for every point on the boundary, where L and C depends on $n, \epsilon, \rho, \mu, \hat{\mu}, \inf u, \|f\|_{C^1}$ and $\inf f$.

(2) We will show that Δu is bounded. The follow proof is for both cases (a) and (b), while the number C is understood as a constant depending on $n, r, \epsilon, \mu, \|f\|_{C^2}$ and $\inf f$ for case (a), and $n, r, \epsilon, \rho, \mu, \hat{\mu}, \inf u, \|f\|_{C^2}$ and $\inf f$ for case (b), respectively.

Let $H = \eta(\Delta u + |\nabla u|^2 + n\mu u_n)e^{ax_n}$ where a is some number decided later. Denote $r^2 := \sum_i x_i^2$. Let $\eta(r)$ be a cutoff function such that $0 \leq \eta \leq 1$, $\eta = 1$ in $\overline{B}_{\frac{r}{2}}^+$ and $\eta = 0$ outside \overline{B}_r^+ , and also $|\nabla \eta| < C\frac{\eta^{\frac{1}{2}}}{r}$ and $|\nabla^2 \eta| < \frac{C}{r^2}$. By (8), Δu is positive. Without loss of generality, we may assume $r = 1$ and

$$K = \Delta u + |\nabla u|^2 + n\mu u_n \gg 1.$$

At a boundary point, note that $\eta_n = 0$ because $\eta = \eta(r)$. Differentiating H in the normal direction and using (5) and (6) gives

$$\begin{aligned} H_n &= \eta_n(Ke^{ax_n}) + \eta(K_n + aK)e^{ax_n} = \eta(K_n + aK)e^{ax_n} \\ &\geq \eta((u_{nnn} + (2\mu - \hat{\mu}e^{-u})K + (-3\mu + \hat{\mu}e^{-u})u_{nn} - C) + aK)e^{ax_n}. \end{aligned}$$

By (8) and the inequalities (10) and (12) for cases (a) and (b), respectively, we obtain

$$H_n \geq \eta(-L\Delta u + (2\mu - \hat{\mu}e^{-u})K - C + aK)e^{ax_n} > 0$$

for $a > L - 2\mu + \hat{\mu} \sup e^{-u} + 1$. Thus, H increases toward the interior and the maximum of H must happen at some point x_0 in the interior.

At the maximal point x_0 , we have

$$H_i = \eta_i(Ke^{ax_n}) + \eta e^{ax_n}(K_i + aK\delta_{in}) = 0, \quad (13)$$

and

$$\begin{aligned} H_{ij} &= \eta_{ij}(K + e^{ax_n}) + \eta_i(Ke^{ax_n})_j + \eta_j(Ke^{ax_n})_i + \eta(Ke^{ax_n})_{ij} \\ &= (\eta_{ij} - 2\eta^{-1}\eta_i\eta_j)Ke^{ax_n} + \eta e^{ax_n}(K_{ij} + aK_i\delta_{jn} + aK_j\delta_{in} + a^2K\delta_{in}\delta_{jn}) \end{aligned}$$

is negative semi-definite, where in the second equality we have used (13). Using the positivity of F^{ij} and (13) again to replace K_i and K_j , we get

$$\begin{aligned} 0 &\geq F^{ij}H_{ij}e^{-ax_n} = F^{ij}((\eta_{ij} - 2\eta^{-1}\eta_i\eta_j)K + \eta(K_{ij} - a\frac{\eta_i}{\eta}K\delta_{jn} - a\frac{\eta_j}{\eta}K\delta_{in} - a^2K\delta_{in}\delta_{jn})) \\ &\geq \eta F^{ij}K_{ij} - C \sum_i F^{ii}K, \end{aligned} \quad (14)$$

where we use conditions on η in the inequality. By direct computations,

$$F^{ij}K_{ij} = F^{ij}u_{llij} + F^{ij}(2u_{li}u_{lj} + 2u_{il}u_{lj} + n\mu u_{nij}) = I + II.$$

For I, notice that

$$\hat{A}_{ij,l} = u_{ijll} + 2u_{il}u_{jl} + u_iu_{jll} + u_ju_{ill} - (u_ku_{kl} + u_{kl}^2)g_{ij}.$$

Then

$$I = F^{ij}(\hat{A}_{ij,ll} - 2u_{li}u_{lj} - 2u_{il}u_{lj} + (u_{lk}^2 + u_k u_{kl})g_{ij}),$$

where $F^{ij}(u_i u_{jl}) = F^{ij}(u_j u_{il})$ because F^{ij} is symmetric. Now using (13) to replace u_{li} and u_{kl} yields

$$\begin{aligned} I = & F^{ij} \hat{A}_{ij,ll} + F^{ij}(-2u_{li}u_{lj} - 2u_j(-2u_l u_{li} - n\mu u_{ni} - \frac{\eta_i}{\eta}K - aK\delta_{in}) \\ & + (|\nabla^2 u|^2 + u_k(-2u_l u_{lk} - n\mu u_{nk} - \frac{\eta_k}{\eta}K - aK\delta_{kn}))g_{ij}). \end{aligned}$$

By (8) and the conditions on η , we have

$$\begin{aligned} I \geq & F^{ij} \hat{A}_{ij,ll} + F^{ij}(-2u_{li}u_{lj} + 4u_j u_l u_{li} + (|\nabla^2 u|^2 - 2u_k u_l u_{lk})g_{ij}) \\ & - C \sum_i F^{ii} \eta^{-\frac{1}{2}} (1 + |\nabla^2 u|^{\frac{3}{2}}). \end{aligned}$$

For II, we use the formula

$$\hat{A}_{ij,l} = u_{ijl} + u_i u_{jl} + u_j u_{il} - u_k u_{kl} g_{ij}$$

to obtain

$$\begin{aligned} II = & F^{ij}(2u_{li}u_{lj} + 2u_l u_{ijl} + n\mu u_{ijn}) = F^{ij}(2u_{li}u_{lj} + 2u_l(\hat{A}_{ij,l} - 2u_i u_{jl} + u_k u_{kl} g_{ij}) \\ & + n\mu(\hat{A}_{ij,n} - 2u_i u_{jn} + u_k u_{kn} g_{ij})) \\ \geq & F^{ij}(2u_{li}u_{lj} + 2u_l \hat{A}_{ij,l} - 4u_i u_{jl} u_j + 2u_k u_{kl} u_l g_{ij} + n\mu \hat{A}_{ij,n}) - C \sum_i F^{ii} |\nabla^2 u|^{\frac{3}{2}}. \end{aligned}$$

Combining I and II together, we find that

$$\begin{aligned} F^{ij} K_{ij} \geq & F^{ij} \hat{A}_{ij,ll} + F^{ij}(-2u_{li}u_{lj} + 4u_j u_l u_{li} + (|\nabla^2 u|^2 - 2u_k u_l u_{lk})g_{ij}) \\ & + F^{ij}(2u_{li}u_{lj} + 2u_l \hat{A}_{ij,l} - 4u_i u_{jl} u_j + 2u_k u_{kl} u_l g_{ij} + n\mu \hat{A}_{ij,n}) \\ & - C \sum_i F^{ii} \eta^{-\frac{1}{2}} (1 + |\nabla^2 u|^{\frac{3}{2}}). \end{aligned}$$

Here is the key step of the proof. Three terms from I cancel out three terms from II. Thus, after the cancellations we arrive at

$$\begin{aligned} F^{ij} K_{ij} \geq & F^{ij} \hat{A}_{ij,ll} + F^{ij} |\nabla^2 u|^2 g_{ij} + F^{ij} (2u_l \hat{A}_{ij,l} + n\mu \hat{A}_{ij,n}) \\ & - C \sum_i F^{ii} \eta^{-\frac{1}{2}} (1 + |\nabla^2 u|^{\frac{3}{2}}). \end{aligned}$$

Now returning to (14), applying η on both sides produces

$$\begin{aligned} 0 \geq & \eta^2 F^{ij} K_{ij} - C \sum_i F^{ii} \eta K \\ \geq & \eta^2 F^{ij} \hat{A}_{ij,ll} + \eta^2 F^{ij} |\nabla^2 u|^2 g_{ij} + \eta^2 F^{ij} (2u_l \hat{A}_{ij,l} + n\mu \hat{A}_{ij,n}) \\ & - C \sum_i F^{ii} (1 + \eta^{\frac{3}{2}} |\nabla^2 u|^{\frac{3}{2}}). \end{aligned}$$

By the concavity of F and Lemma 1(a), we have $F^{ij}\hat{A}_{ij,l} \geq (F^{ij}\hat{A}_{ij})_l = (fe^{-2u})_l$. Hence,

$$\begin{aligned} 0 &\geq \eta^2 \sum_i F^{ii} |\nabla^2 u|^2 + \eta^2 (fe^{-2u})_l + 2\eta^2 u_l (fe^{-2u})_l + n\mu\eta^2 (fe^{-2u})_n \\ &\quad - C \sum_i F^{ii} (1 + \eta^{\frac{3}{2}} |\nabla^2 u|^{\frac{3}{2}}) \\ &\geq \sum_i F^{ii} (\eta^2 |\nabla^2 u|^2 - C - C\eta |\nabla^2 u| - C\eta^{\frac{3}{2}} |\nabla^2 u|^{\frac{3}{2}}). \end{aligned}$$

This gives $(\eta |\nabla^2 u|)(x_0) \leq C$. Hence, for $x \in \overline{B}_{\frac{r}{2}}^+$, we have that $H = (\Delta u + |\nabla u|^2 + n\mu u_n)e^{ax_n}$ is bounded. Thus, Δu is bounded. By (8), $|\nabla u|$ is also bounded.

(3) To get the Hessian bounds, for case (b) it follows immediately by the fact that if $\Gamma_2^+ \subset \Gamma$, then $|u_{ij}| \leq C\Delta u$. As for case (a), note that from (2) above we have $\eta\Delta u < C$ and $\eta|\nabla u|^2 < C$. Consider the maximum of $\eta(\nabla^2 u + du \otimes du + \mu u_n g)e^{ax_n}$ over the set $(x, \xi) \in (B_1^+, S^n)$. We will show that at the maximum, x can not belong to the boundary. If ξ is in the tangential direction, without loss of generality, we can assume ξ is in e_1 direction. We have

$$\begin{aligned} (\eta(u_{11} + u_1^2 + \mu u_n)e^{ax_n})_n &= \eta(u_{11n} + 2u_1 u_{1n} + \mu u_{nn} + a(u_{11} + u_1^2 + \mu u_n))e^{ax_n} \\ &= \eta e^{ax_n} ((2\mu + a)(u_{11} + u_1^2 + \mu u_n) + \mu^3) > 0 \end{aligned}$$

for $a > -2\mu + 1$. If ξ is in the normal direction, we first have that $\Delta u \leq n(u_{nn} + \mu^2) \leq nu_{nn} + C$. By (10), we obtain

$$\begin{aligned} (\eta(u_{nn} + u_n^2 + \mu u_n)e^{ax_n})_n &= \eta(u_{nnn} - \mu u_{nn} + a u_{nn})e^{ax_n} \\ &\geq \eta e^{ax_n} (-L\Delta u + 2\mu u_{nn} + a u_{nn} - C) \\ &\geq \eta e^{ax_n} (-nL u_{nn} + 2\mu u_{nn} + a u_{nn} - C) > 0 \end{aligned}$$

for $a > nL - 2\mu + 1$. Thus, we conclude that at the maximum, x must be in the interior. We then perform similar computations as before using the inequality $\eta|\nabla u|^2 < C$ to get the Hessian bounds. We omit the details here. \square

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